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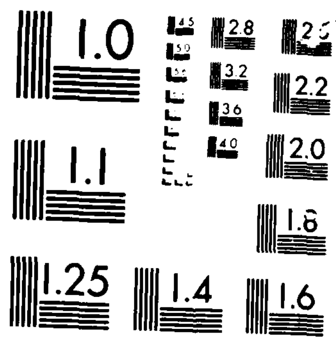

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TOUGHNESS AND MATCHING EXTENSION IN GRAPHS

by

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1. Introduction and Terminology

All graphs in this paper will be finite and connected and will have no loops or parallel lines.

Let n and p be positive integers with $n \leq (p-2)/2$ and let G be a graph with p points having a perfect matching. Graph G is said to be **n-extendable** if every matching of size n in G extends to a perfect matching.

The concept of n-extendability for bipartite graphs was studied by Hetyei (1964). But the study of the more general family of n-extendable graphs which are not necessarily bipartite seems to have even earlier roots. In the late 1950's, Kotzig (1959a, 1959b, 1960) began to develop a decomposition theory for graphs with perfect matchings, but unfortunately these papers did not receive the attention that they deserve, due to the fact that they were written in Slovak. In the early 1960's, the study of decompositions of graphs in terms of their maximum matchings was begun by Gallai (1963, 1964) and independently by Edmonds (1965). One of the degenerate cases of their theory for *maximum* matchings, however, arises when the graphs in question have *perfect* matchings.

Motivated by these results of Kotzig, Hetyei, Gallai and Edmonds, Lovász (1972) extended and refined the canonical decompositions already extant.

In this same paper, Lovász also introduced the concept of a *bicritical* graph. A graph G is said to be **bicritical** if $G-u-v$ has a perfect matching for every pair of distinct points u and v in $V(G)$. In the last ten years or so, the earlier work on decompositions of graphs in terms of their matchings has evolved further (see Lovász and Plummer (1986)) and today much attention continues to be focused upon the structure of bicritical graphs which are, in addition, 3-connected. Such graphs

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have been christened **bricks**. (See, for example, the paper by Edmonds, Lovász and Pulleyblank (1982) and that of Lovász (1986).)

But what is the connection between n-extendability and bicriticality? One of the results presented in Plummer (1980) states that every 2-extendable graph is either bipartite or is a brick. (The reader should convince himself immediately that these two classes of graphs are disjoint.) Motivated by this result, the author has continued to study properties of n-extendable graphs (see (1985, 1986a, 1986b and 1986c)).

Let S be a point cutset in graph G and let $c(G-S)$ denote the number of components in $G-S$. Then, if G is not complete, the **toughness** of G is defined to be min $\frac{|S|}{c(G-S)}$ where the minimum is taken over all point cutsets S of G, whereas we define the toughness of K_n to be $+\infty$ for all n. We denote the toughness of G by **tough(G)**. We will also say that graph G is k-**tough** if tough$(G) \geq k$. This parameter was introduced by Chvátal (1973a, 1973b) who was initially motivated by studies about Hamiltonian cycles in graphs. He noted, however, in (1973a) that every 1-tough graph with an even number of points has a perfect matching.

A generalization of both the concepts of Hamiltonian cycle and perfect matching is the idea of a *k-factor* of a graph. A k-**factor** of a graph G is a spanning subgraph of G which is regular of degree k. Thus a perfect matching is just a 1-factor and a Hamiltonian cycle is just a connected 2-factor. Chvátal conjectured in (1973a) that if G is any graph on p points and if k is a positive integer such that G is k-tough and kp is even, then G has a k-factor. This conjecture has only recently been settled in the affirmative by Enomoto, Jackson, Katerinis and A. Saito (1985).

In the present paper, we wish to treat some relationships between toughness of a graph and the n-extendability of the graph. In the next section we will prove two results. The first says essentially that if a graph has sufficiently high toughness (and has an even number of points) then it must be n-extendable. The second result applies to graphs with toughness less than one and presents an upper bound on the value of n for which such a graph can be n-extendable.

In the final section, we compare and contrast these results with the n-factor results of Enomoto, Jackson, Katerinis and A. Saito.

Any graph terminology used, but not defined, in this paper may be found either in Bondy and Murty (1977) or Lovász and Plummer (1986).

2. Two results on toughness and n-extendability

In addition to the theorem of the author (1980) mentioned in the Introduction, there are two other results proved in that paper which we shall use here and hence we begin by stating them without proof.

1980A. THEOREM. *If $n \geq 2$ and G is n-extendable, then G is also $(n-1)$-extendable.* ■

1980B. THEOREM. *If G is n-extendable, then G is $(n+1)$-connected.* ■

Our first result of the present paper follows in a straightforward way via Tutte's classical theorem characterizing graphs with perfect matchings.

2.1. THEOREM. *Suppose that G is a graph with $p = |V(G)|$ points with p even. Let n be a positive integer with $p \geq 2n+2$. Then if* tough$(G) > n$*, graph G is n-extendable. Moreover, this lower bound on* tough(G) *is sharp for all n.*

PROOF. First suppose that $n = 1$. Note that since tough$(G) \geq 1$, graph G has a perfect matching by Tutte's Theorem on perfect matchings.

Now suppose that for some line $e = xy \in E(G)$, line e lies in no perfect matching for G. Thus if $G' = G - x - y$, by the above-mentioned theorem of Tutte there is a set $S' \subsetneq V(G')$ with $|S'| < c_o(G' - S')$. (Note that here $c_o(G' - S')$ denotes the number of components of $G' - S'$ which have an odd number of points.) But then by parity, $|S'| \leq c_o(G' - S') - 2$.

Now let $S_0 = S' \cup \{x, y\}$. Since G has a perfect matching, it follows that $c_o(G - S_0) \leq |S_0| = |S'| + 2 \leq c_o(G' - S')$. But $G - S_0 = G' - S'$ and so equality holds throughout and in particular, $c_o(G - S_0) = |S_0|$. (See Figure 1.)

But now

$$\text{tough}(G) = \min_{S \subsetneq V(G)} \frac{|S|}{c(G-S)} \leq \frac{|S_0|}{c(G-S_0)} \leq \frac{|S_0|}{c_o(G-S_0)} = 1,$$

contradicting the hypothesis of this theorem. So the desired conclusion holds when $n = 1$.

Now suppose $n \geq 2$, and assume that G is not n-extendable. Let $M = \{e_1, \ldots, e_n\}$ be a matching of size n which does not extend to a perfect matching. Denote $e_i = x_i y_i$ for $i = 1, \ldots, n$. Let $G_1 = G - x_1 - \cdots - x_n - y_1 - \cdots - y_n$. Hence G_1 has no perfect matching

FIGURE 1.

and thus by Tutte's Theorem, there is a set $S_1 \subseteq V(G_1)$ such that $|S_1| < c_o(G_1 - S_1)$. Hence by parity, $|S_1| \leq c_o(G_1 - S_1) - 2$. (Note that S_1 might be empty.)

Now $G_2 = G - x_n - y_n$ has a perfect matching since we have already proved that G is 1-extendable. Let $S_2 = S_1 \cup \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$ and note that once again by Tutte's Theorem, $c_o(G_2 - S_2) \leq |S_2| = |S_1| + 2n - 2 \leq c_o(G_1 - S_1) + 2n - 4$. But $G_2 - S_2 = G_1 - S_1$ and so it follows that $c_o(G_2 - S_2) \geq |S_1| + 2$.

Now let $S_3 = S_1 \cup \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Then, since $G - S_3 = G_2 - S_2$, we have

$$\begin{aligned}
\text{tough}(G) &\leq \min_{S \subseteq V(G)} \frac{|S|}{c(G-S)} \leq \frac{|S_3|}{c(G-S_3)} = \frac{|S_3|}{c(G_2-S_2)} \\
&\leq \frac{|S_3|}{c_o(G_2-S_2)} = \frac{|S_1|+2n}{c_o(G_2-S_2)} \\
&\leq \frac{|S_1|+2n}{|S_1|+2} \leq \frac{n|S_1|+2n}{|S_1|+2} = n,
\end{aligned}$$

again a contradiction of the hypothesis.

It remains only to exhibit extremal graphs for each value of $n \geq 1$. For each positive integer $n \geq 1$, define graph H_n as follows. Let $N = \{e_1, \ldots, e_n\}$ be a set of n independent lines. Join each of the $2n$ points of N to each point of two disjoint copies of the complete graph K_{2n+1}. (See Figure 2.) Then $|V(H_n)| = 6n + 2$ and it is easy to see that

FIGURE 2. The extremal family $\{H_n\}_{i=1}^{\infty}$

tough$(H_n) = n$. However, the matching N does not extend to a perfect matching. ■

Now let us begin to think of some type of converse to the above result. Let us remark at the outset that if a graph G is n-extendable, *there is no lower bound on the toughness of* G. To see this, we construct the following family of graphs. Let n and k be any two positive integers. Let S be a set of $2n$ independent points and let graph $J(n,k)$ be constructed by joining each point of set S to both endpoints of each of $2n+k$

FIGURE 3. The extremal family $\{J(n,k)\}_{n=1,k=1}^{\infty}$

independent lines. (See Figure 3.)

It is easy to verify that $J(n,k)$ is n-extendable for every value of k. Clearly, $\text{tough}(J(n,k)) \leq 2n/(2n+k)$, and hence $\text{tough}(J(n,k)) \to 0$ as $k \to \infty$. Of course the number of points in graph $J(n,k)$ is quite large and it makes sense to amend our search for some type of converse to Theorem 2.1 as follows. Again letting $p = |V(G)|$, we may ask if there is a function $f(p)$ such that if graph G is $f(p)$-extendable, then G is, say, 1-tough. The next result shows that the answer to this question is "yes".

2.2. THEOREM. *Let G be a graph with p points and let n be a positive integer. Suppose that G is n-extendable, but that* $\text{tough}(G) < 1$. *Then $n \leq \lfloor \frac{p-2}{6} \rfloor$ and this bound is sharp for each n.*

PROOF. Since $\text{tough}(G) < 1$, there is a cutset S in G such that $G - S$ has more than $|S| = s$ components. Note that by Theorem 1980B, $s \geq n+1 \geq 2$. Let the components of $G - S$ be $C_1, \ldots, C_{s+r}$, where $r \geq 1$.

Note that $G - S$ must have at least one even component, for if not, by Tutte's Theorem, G could not have a perfect matching, contradicting the hypothesis of the present theorem. So suppose that component C_1 is even and hence $|V(C_1)| \geq 2$. Since $n \geq 1$, G is 2-connected by Theorem 1980B, and hence there exists a line e_1 joining a point of C_1 to a point of S. By hypothesis, G is n-extendable and $n \geq 1$, so by Theorem 1980A, G is 1-extendable. So extend line e_1 to a perfect matching F_1 of G and note that by parity, F_1 matches at least two points of component C_1

into set S. It then follows that, in fact, $G - S$ must have at least 3 even components.

Claim 1. $G - S$ has at least n even components.

If $1 \leq n \leq 3$, we are done. So we may suppose that $n \geq 4$.

Suppose, to the contrary, that $G - S$ has t even components, where $3 \leq t \leq n - 1$. Relabeling these components if necessary, suppose that $C_1, \ldots, C_t$ are the even components of $G - S$. (Recall that altogether, $G - S$ has $s + r \geq s + 1 \geq n + 2$ components.)

We now construct a matching which contains two lines joining each of the components $C_1, \ldots, C_t$ to different points of S. Let e_1 be any line joining C_1 to a point u_1 in S, relabeling the points of S if necessary. Now if all lines between S and C_2 are incident with point u_1, it then follows that u_1 is a cutpoint of G, a contradiction of the fact that G is 2-connected. Hence we can match points of C_1 and C_2 via lines e_1 and e_2 to distinct points u_1 and u_2 of S say, where once again we relable the points of S if necessary.

Recall that $n \geq 4$. Suppose further that $C_1, \ldots, C_q$, $q < n$, have been matched into S to points $u_1, \ldots, u_q$ respectively. If we cannot match a point of $V(C_{q+1})$ into S at a point different from $u_1, \ldots, u_q$, then $\{u_1, \ldots, u_q\}$ is a cutset of G; that is, it separates C_{q+1} from all the other components of $G - S$. Hence $\kappa(G) \leq q < n$, contradicting the fact that (by Theorem 1980B) graph G is $(n+1)$-connected. Thus we have the matching of size n that we seek. Call it M_1.

Extend matching M_1 to a perfect matching F_2 of G. By parity, for each even component $C_1, \ldots, C_t$, matching F_2 must match at least one point to S other than that matched by M_1. Without loss of generality, let us suppose that a point of C_1 is matched to point $u_{n+1}, \ldots$, and that a point of C_t is matched to point u_{n+t}. (See Figure 4.)

But now each of the odd components $C_{n+1}, \ldots, C_{s+r}$ must contain at least one point which is matched by perfect matching F_2 into the set $\{u_{n+t+1}, \ldots, u_s\}$ of S. Thus it follows that $s + r - n \leq s - (n + t)$ and so $r \leq -t < 0$, a contradiction and Claim 1 is proved.

Finally, we prove

Claim 2. Graph $G - S$ has at least $2n + r$ even components.

By Claim 1, graph $G - S$ has at least n even components. Relabeling if necessary, suppose that they are $C_1, \ldots, C_n, \ldots, C_{n+\gamma}$. Just as in the proof of Claim 1, since G is n-connected, we can find a matching M_2 which joins exactly one point of component C_i to a point u_i in S for $i = 1, \ldots, n$, where yet again, we renumber points in S if necessary.

Since G is n-extendable, let us extend matching M_2 to a perfect

FIGURE 4.

matching F_3 of G. By parity, each of $C_1, \ldots, C_n$ has at least 2 points matched into S by F_3. So relabeling again if need be, assume that F_3 also matches a point u_{n+i} to a point in component C_i for $i = 1, \ldots, n$. (In particular at this point, we now know that $|S| = s \geq 2n$.)

Thus F_3 must match the remaining $s-2n$ points of S (if any) to some $s-2n$ points in $\bigcup_{i=1}^{s+r} V(C_i)$. So among the components $C_{n+1}, \ldots, C_{s+r}$, at least $(s+r-n)-(s-2n) = n+r$ must be even. These components, together with $C_1, \ldots, C_n$ give the $2n+r$ even components of $G-S$ as claimed.

Now we have

$$|V(G)| = p = |S| + |V(C_1)| + \cdots + |V(C_{s+r})|$$

$$\geq s + 2(2n+r) + (s + r - (2n+r))$$

$$= s + 4n + 2r + s - 2n = 2s + 2n + 2r \geq 6n + 2r \geq 6n + 2.$$

So $n \leq (p-2)/6$ and since n is an integer, $n \leq \lfloor \frac{p-2}{6} \rfloor$.

To show that the bound is sharp for all $n \geq 1$ consider the infinite family of graphs L_n where $L_n = J(n,1)$ and $J(n,1)$ is shown in Figure 3. Note that $p = |V(H_n)| = 2n+2(2n+1) = 6n+2$ and hence $n = (p-2)/6$ and it is easy to check that graph H_n is n-extendable, but not $(n+1)$-extendable. ∎

Of course, Theorem 2.2 can be restated as follows: if graph G is $(\lfloor \frac{p-2}{6} \rfloor) + 1$-extendable, then G is 1-tough.

3. Comparisons with an n-factor theorem

Enomoto, Jackson, Katerinis and A. Saito (1985) have proved the following result.

1985. THEOREM. *Let G be a graph with at least $n+1$ points and suppose* $\text{tough}(G) \geq n$. *Then, if $n|V(G)|$ is even, G has an n-factor.*

This theorem answers in the affirmative a conjecture of Chvátal. In order to properly compare the conclusion of this result with that of our Theorem 2.1, let us try to state each result in as parallel a fashion as possible. Of course, if we were to define a graph to be "0-extendable" if it had a perfect matching, the two conclusions would say exactly the same thing when $n = 1$.

Now suppose $n \geq 2$ and consider the following two statements; the first being the result of Theorem 1985 and the second, our Theorem 2.1.

(A) $\text{tough}(G) \geq n \Rightarrow G$ has an n-factor.

(B) $\text{tough}(G) \geq n \Rightarrow G$ is $(n-1)$-extendable.

We claim that the two conclusions are independent, in that neither implies the other.

First consider the family of graphs $J(n,1)$ already discussed above. Suppose $n \geq 2$. We already know that graph $J(n,1)$ is n-extendable. Hence by Theorem 1980A it is also $(n-1)$-extendable. We claim it has no n-factor.

Suppose, to the contrary, that $J(n,1)$ does have an n-factor, F. Then factor F must send $2n^2$ lines from set S to $G-S$. But each point of $G-S$ must send at least $n-1$ lines to set S and hence we have at least $(n-1)(4n+2) = 4n^2 - 2n - 2$ lines of factor F from $G-S$ to S. But then $2n^2 \geq 4n^2 - 2n - 2$ and it follows that $n = 1$, a contradiction.

Finally, consider the infinite family of graphs $\{M_n\}_{n=2}^{\infty}$ constructed as follows. Graph M_n is formed by taking two copies of the complete graph K_{n+1} and joining corresponding points of the two copies with a perfect matching. (This is, of course, just the *prism* over K_{n+1}.) Graph M_n clearly has an n-factor consisting of precisely two components; namely, the two copies of K_{n+1}. On the other hand, we claim that graph M_n is not $(n-1)$-extendable. Let the points of the "top" K_{n+1} be $u_1, \ldots, u_{n+1}$ and the corresponding points of the "bottom" K_{n+1} be $v_1, \ldots, v_{n+1}$. (That is, u_iv_i, $i = 1, \ldots, n+1$, is the perfect matching joining the top and bottom.)

In order to prove our assertion, let us consider the cases for n odd and even separately.

FIGURE 5. Extremal graphs M_4 and M_5

First suppose n is odd. Select the matching M_o consisting of $u_1v_1, u_2u_n, u_3u_{n-1}, \ldots$ together with line v_nv_{n+1}. Clearly M_o is a matching of size $n-1$, but it cannot be extended to a perfect matching for point u_{n+1} could never be matched in such a extension.

Now suppose n is even. First let us suppose also that $n \geq 4$. In this case, select the matching M_e to consist of $u_1u_n, u_2u_{n-1}, \ldots$ together with line v_1v_{n+1}. Then matching M_e has size $n-1$, but it cannot be extended to a perfect matching since point u_{n+1} could never be matched.

Finally, suppose $n = 2$. Let M_2 be the graph $K_4 - e$ for any line e in K_4. Clearly M_2 is not 1-extendable, but it has a 2-factor.

We show the graphs M_4 and M_5 in Figure 5.

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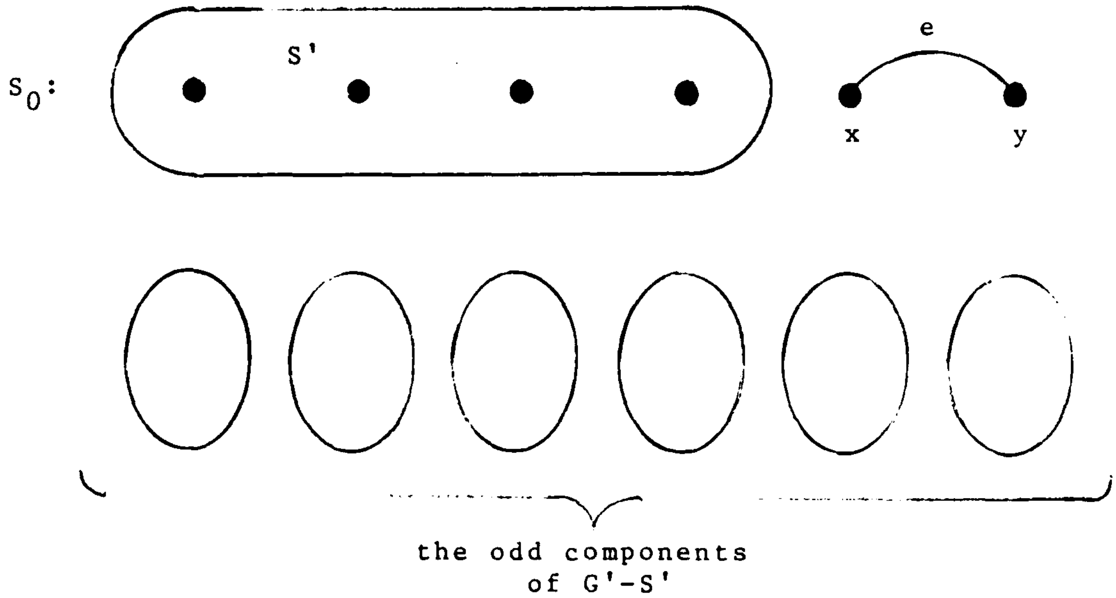


Figure 1

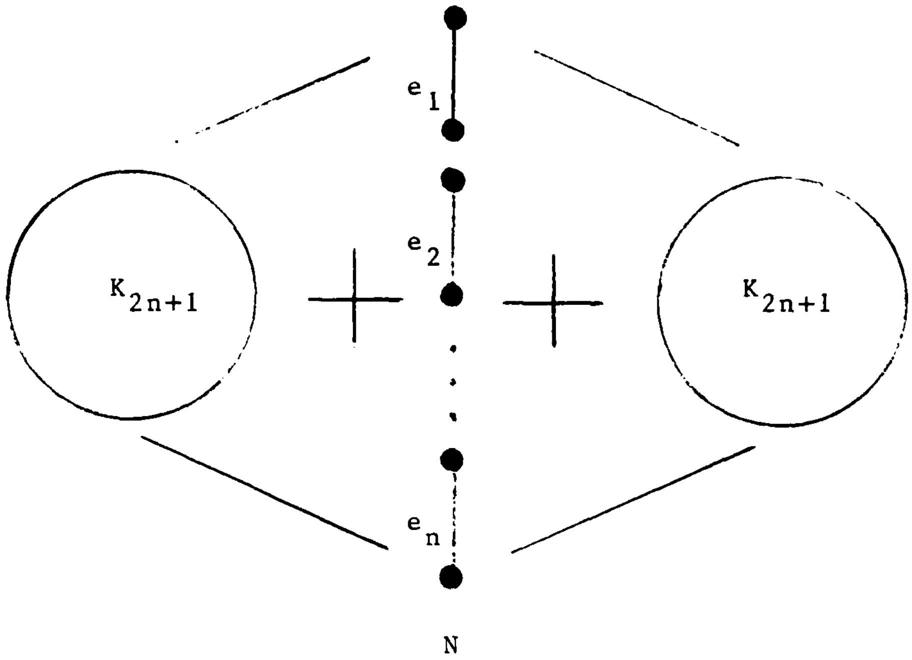


Figure 2

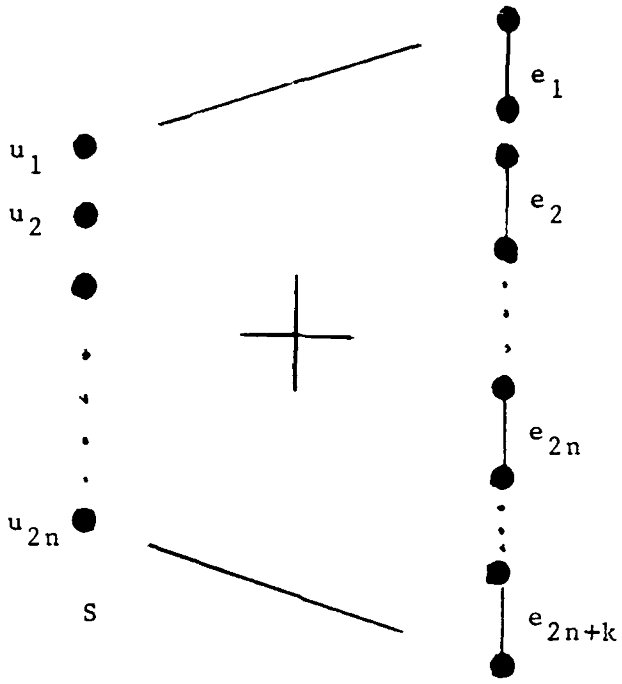


Figure 3

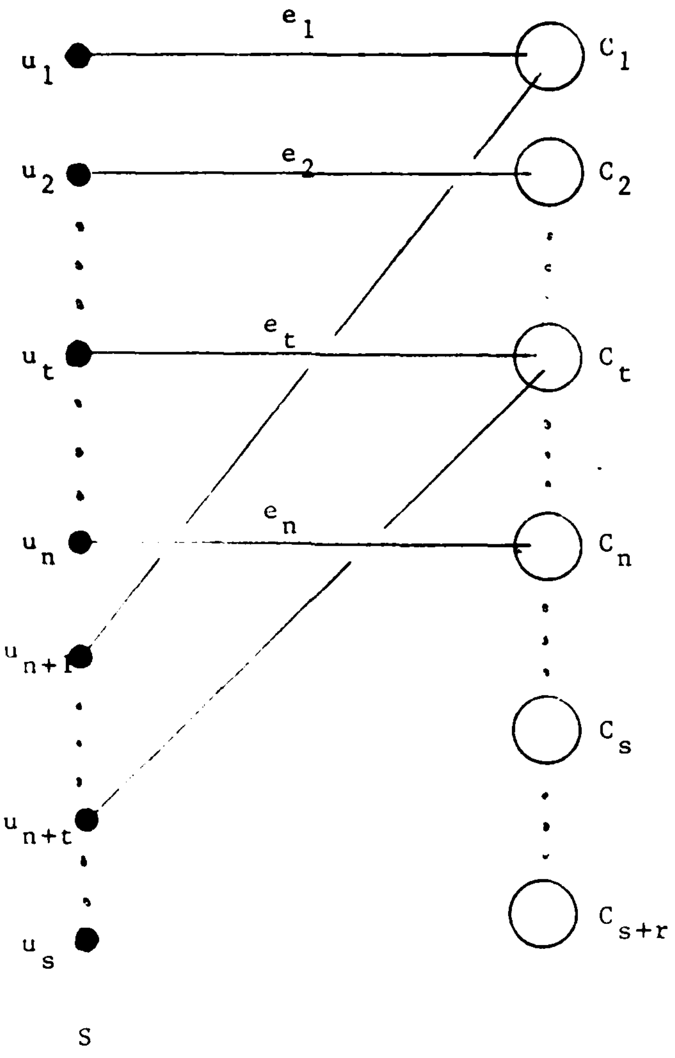
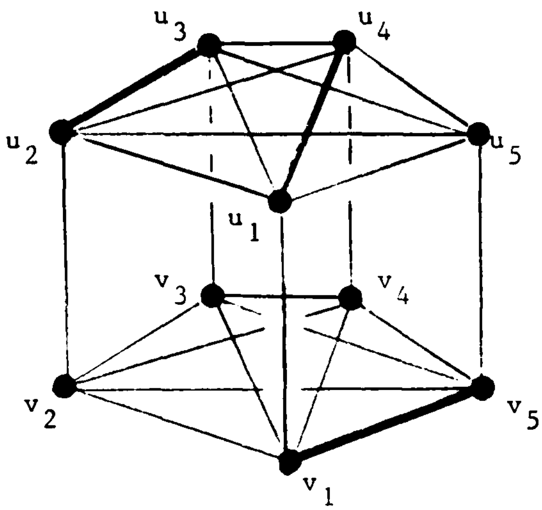
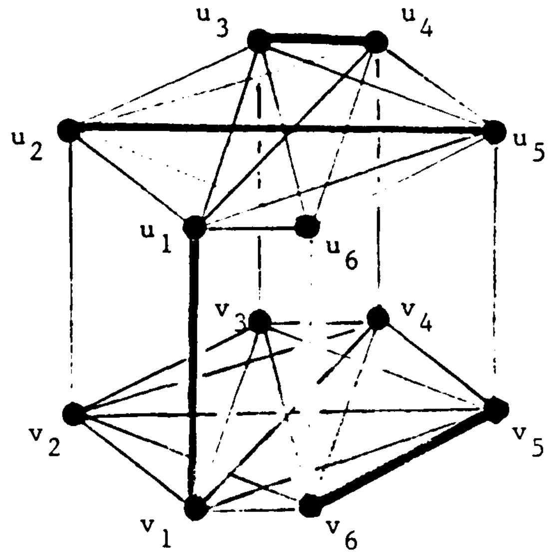


Figure 4



M_4



M_5

Figure 5

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